AE265 – 2000 – MIDTERM ANSWER SHEET
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1. Using the Taylor Table approach on the finite difference approximation of the 1^{st} derivative

$$\left(\frac{\partial u}{\partial x}\right)_j + c\left(\frac{\partial u}{\partial x}\right)_{j+\beta} = \left(a u_j + b u_{j+1}\right)/\Delta x$$

(a) Find the coefficients a, b, and c in terms of β which minimize the error er_t. (Points:4)
(HINT: $u_{j+\beta} = u_j + \beta\Delta x\left(\frac{\partial u}{\partial x}\right)_j + \frac{1}{2!}(\beta\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \frac{1}{3!}(\beta\Delta x)^3\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \cdots$*)*

(b) Find the resulting expression for er_t, in terms of β and find the value of β which *further* minimizes the error. (Points:4)

ANSWER Problem #1 From the Taylor table

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$
$c \cdot \Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_{j+\beta}$		c	$c \cdot (+\beta) \cdot \frac{1}{1}$	$c \cdot (+\beta)^2 \cdot \frac{1}{2}$	$c \cdot (+\beta)^3 \cdot \frac{1}{6}$
$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$		1			
$-b \cdot u_{j+1}$	$-b$	$-b \cdot (1) \cdot \frac{1}{1}$	$-b \cdot (1)^2 \cdot \frac{1}{2}$	$-b \cdot (1)^3 \cdot \frac{1}{6}$	$-b \cdot (1)^4 \cdot \frac{1}{24}$
$-a \cdot u_j$	$-a$				

the following equation has been constructed to maximize the order of accuracy

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -\frac{1}{2} & \beta \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

This has the solution

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{-2\beta}{2\beta-1} \\ \frac{2\beta}{2\beta-1} \\ \frac{1}{2\beta-1} \end{bmatrix}$$

The Taylor series error of this difference scheme is

$$\begin{aligned} er_t &= \left(-b\frac{1}{6} + c\beta^2\frac{1}{2}\right)\Delta x^2\left(\frac{\partial^3 u}{\partial x^3}\right)_j \\ &= \frac{\beta(3\beta-2)}{6(2\beta-1)}\Delta x^2\left(\frac{\partial^3 u}{\partial x^3}\right)_j \end{aligned}$$

This shows that the scheme is second order for arbitrary β.

To further minimize the error, let $\beta = \frac{2}{3}$, thereby eliminating the above term and forcing the error out to the next term

$$\begin{aligned} er_t &= \left(-b\frac{1}{24} + c(\beta)^3\frac{1}{6}\right)\Delta x^3\left(\frac{\partial^4 u}{\partial x^4}\right)_j \\ &= \frac{-1}{54}\Delta x^3\left(\frac{\partial^4 u}{\partial x^4}\right)_j \end{aligned}$$

Now a third order method.

2. Find the expression for the modified wave number of the scheme in terms of Δx and k. Cast the result in terms of $sin's$ and $cos's$ and where indicated use series expansion to identify the accuracy of the scheme.

   (a) $(\delta_x u)_j = (u_{j-2} - 4u_{j-1} + 4u_{j+1} - u_{j+2}) / (4\Delta x)$ and identify the accuracy of the scheme. (Points:4)

   (b) $(\delta_{xxxx} u)_j = (u_{j-2} - 4u_{j-1} + 6u_j - 4u_{j+1} + u_{j+2}) / \Delta x^4$ and identify the accuracy of the scheme. (Points:4)
   (HINT: $\delta_{xxxx} e^{ikj\Delta x} = (k^*)^4 e^{ikj\Delta x}$, find $(k^*)^4 = k^4 + O(\Delta x^p)$, that is, don't try to take the 4th root..)

ANSWER Problem #2a

We apply $u_j = e^{ikj\Delta x}$ to both sides and get

$$\left(ik^* e^{ikj\Delta x}\right) = e^{ikj\Delta x}\left(e^{-2ik\Delta x} - 4e^{-ik\Delta x} + 4e^{+ik\Delta x} - e^{2ik\Delta x}\right)/(4\Delta x)$$

which give us

$$ik^* = i\frac{(4sin(k\Delta x) - sin(2k\Delta x))}{2\Delta x}$$

Expanding the sin function gives

$$k^* = k + \frac{1}{3}k^3\Delta x^2 + \cdots$$

showing a 2nd order approximation to the first derivative.

ANSWER Problem #2b

We apply $u_j = e^{ikj\Delta x}$ to both sides and get

$$\left((k^*)^4 e^{ikj\Delta x}\right) = e^{ikj\Delta x}\left(e^{-2ik\Delta x} - 4e^{-ik\Delta x} + 6 - 4e^{+ik\Delta x} + e^{2ik\Delta x}\right)/\Delta x^4$$

which give us

$$(k^*)^4 = \frac{(6 - 8cos(k\Delta x) + 2cos(2k\Delta x))}{\Delta x^4}$$

Expanding the cos function gives

$$(k^*)^4 = k^4 - \frac{1}{6}k^6\Delta x^2 + \cdots$$

showing a 2nd order approximation to the fourth derivative.

3. Consider the predictor- corrector method

$$\begin{aligned}\tilde{u}_{n+1} &= u_{n-1} + 2h(\tilde{u}')_n \\ u_{n+1} &= u_n + h(\tilde{u}')_{n+1}\end{aligned}$$

applied to the representative equation

$$u' = \lambda u + ae^{\mu t}$$

Note!!!! I have put in more to this than in the midterm. Here I added to particular solution part in the question and answers. To get what was asked for on the Midterm set $a = 0$ and eliminate the $Q(E)$ parts

(a) Identify the characteristic and particular operators as discussed in class, $[P(E)]$ and $\vec{Q}(E)$ and find the characteristic polynomial $P(E)$. (Points:3)

(b) Find the σ's for this method *(HINT: it is a 2 root method).* (Points:2)

(c) Identify the principal and spurious roots and justify your choice. (Points:2)

(d) Find er_λ and identify the order of this method. (Points:2)

(e) Find the particular solution,u_∞. (Optional:Points:1)

(f) Determine the stability of the method, i.e., conditions on λh.(Optional:Points:2)

(**Note**: *The $\sim$ in $(\tilde{u}')_n$ for the predictor step)*

ANSWER Problem #3a

For the predictor-corrector combination

$$\begin{aligned}\tilde{u}_{n+1} &= u_{n-1} + 2h(\tilde{u}')_n \\ u_{n+1} &= u_n + h(\tilde{u}')_{n+1}\end{aligned}$$

Applying the time-marching scheme to the representative equation

$$\frac{du}{dt} = \lambda u + ae^{\mu t}$$

results in the following equation set

$$\begin{aligned}\tilde{u}_{n+1} &= u_{n-1} + 2h\left(\lambda \tilde{u}_n + ae^{\mu hn}\right) \\ u_{n+1} &= u_n + h\left(\lambda \tilde{u}_{n+1} + ae^{\mu h\,(n+1)}\right)\end{aligned}$$

Introducing the difference operator, E, the equation set may be expressed in matrix form as

$$\begin{bmatrix} E - 2\lambda h & -E^{-1} \\ -\lambda h E & E - 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_n \\ u_n \end{bmatrix} = \begin{bmatrix} 2h \\ hE \end{bmatrix} ae^{\mu hn}$$

The results for $[P(E)]$ and $\vec{Q}(E)$ are obviously from the previous equation.

The characteristic polynomial equals the determinant of the matrix

$$P(E) = E^2 - (1 + 2\lambda h)E + \lambda h$$

The particular polynomial, $Q(E)$, for the final family u_n (as opposed to the intermediate family $\tilde{u}_n$) is given by

$$Q(E) = \det \begin{bmatrix} E - 2\lambda h & 2h \\ -\lambda h E & hE \end{bmatrix} = hE^2$$

ANSWER Problem #3b

The characteristic polynomial is

$$P(\sigma) = \sigma^2 - (1 + 2\lambda h)\sigma + \lambda h$$

giving

$$\sigma_{1,2} = \frac{1}{2} + \lambda h \pm \frac{1}{2}\sqrt{1 + 4\lambda^2 h^2}$$

ANSWER Problem #3c

$$\sigma_1 = \frac{1}{2} + \lambda h + \frac{1}{2}\sqrt{1 + 4\lambda^2 h^2}$$

and

$$\sigma_2 = \frac{1}{2} + \lambda h - \frac{1}{2}\sqrt{1 + 4\lambda^2 h^2}$$

An easy way to check for the principal and spurious roots is to let $h = 0$. For the principal root $\sigma = 1$ is consistent with $e^{\lambda h}$ for $h = 0$ and the spurious root will not equal 1. In this case $\sigma_1 = 1$ and $\sigma_2 = 0$ identifying the two types.

ANSWER Problem #3d

Expanding the square root for the principal root

$$\sigma_1 = 1 + \lambda h + \lambda^2 h^2 + \cdots$$

and the transient error is

$$er_\lambda = -\frac{1}{2}(\lambda h)^2 + O(h^3)$$

a first order method.

ANSWER Problem #3e

The exact numerical solution to $u' = \lambda u + ae^{\mu t}$ is then

$$u_n = c_1\sigma_1^n + c_2\sigma_2^n + ae^{\mu h n}\frac{Q(e^{\mu h})}{P(e^{\mu h})}$$

which give us the Particular Solution

$$u_\infty = ae^{\mu h n} \cdot \frac{he^{2\mu h}}{e^{2\mu h} - (1 + 2\lambda h)e^{\mu h} + \lambda h}$$

ANSWER Problem #3f

Determine the stability of the method, i.e., conditions on λh. This is a little hard to do from the definition of the σ roots directly. The basic condition is $|\sigma_1| \leq 1$ and we also have to check the spurious root $|\sigma_2| \leq 1$. Probably the best way to proceed is to plot the σ roots in both the complex-σ and complex-λ planes as in Chapter 7 of the notes. From a matlab program we have

From the complex-λ plane figure one can pick off the stability bound as approximately $|\lambda h| < \frac{2}{3}$. Functional analysis confirms it.

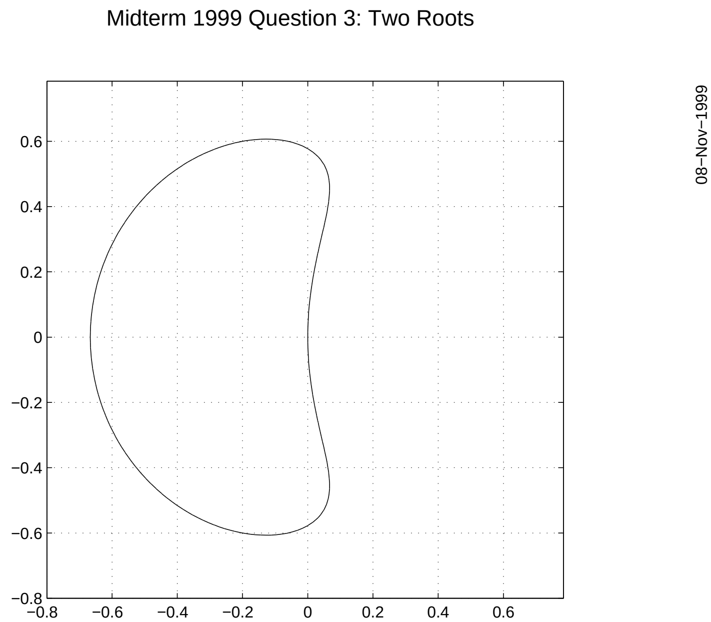

Figure 1: The complex-λ plane plot of $|\sigma| = 1$

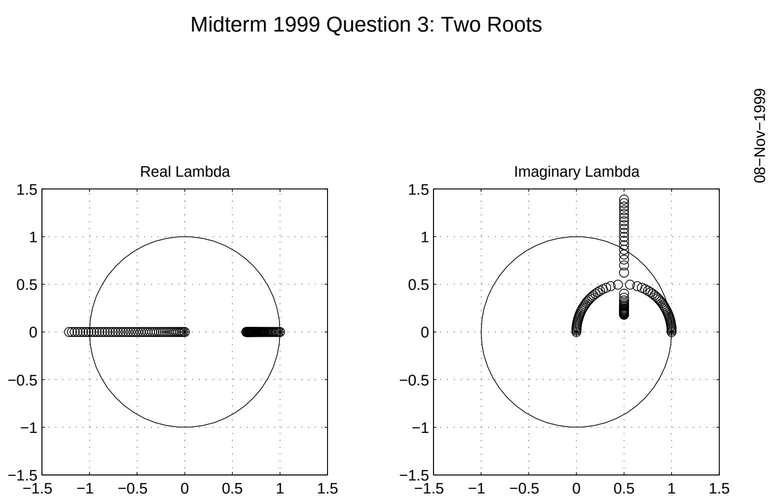

Figure 2: The complex-σ plane plot for Real-λ and Pure Imaginary-Λ